

Black Hole in ADS and Quantum Field Theory

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based on the paper:

G. Maiella and C. Stornaiolo,

“A CFT Description of the BTZ Black Hole: Topology versus Geometry
(or Thermodynamics versus Statistical Mechanics)”,

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Plan of the talk

A) Introduction

B) It is given a detailed analysis of the well-known case $\text{ADS}_3/\text{CFT}_2$.

The geometrical and topological properties are presented.

The exact solution of unitary CFT_2 with central charge $c = 1$ implies exact results on the Black Hole (BH) solution for the ADS_3 space-time, the so-called BTZ (Bañados, Teitelboim, Zanelli, *PRL* **69** (1992), 1849).

C) The peculiarity of BTZ Black Hole is made explicit by showing the topological nature of T_H and its relation with the boundary (quantum) energy $E_C = c/6$.

D) As possible avenue for the extension of $D = 2$ results to any D we analyze the moving mirror analogy of Black Hole and the Verlinde proposal to substitute the quantum anomaly $c/6$ with the Casimir energy E_C .

Both avenues seem to suggest that $S_H = S_{\text{ent}}$ for any $D \geq 2$.

This aspect is under study by G. Maiella and C. Stornaiolo.

A) Introduction

I will present a short review of recent achievements in understanding the origin of the statistical properties versus the thermodynamical ones for Anti De Sitter space–time, ADS_{n+1} from the point of view of the “dual” field theory defined on the boundary of ADS_{n+1} conformal invariant, CFT_n .

We try to convince the audience that the symmetries of Quantum Field Theory at the critical points are, on one side, at the basis of the thermodynamical properties of Black Hole solution, i.e. the Hawking temperature T_H and the entropy S_H ; on the other side the CFT_n in Euclidean time determines the statistical properties of BH.

B) Geometry of the ADS_3 space–time

The ADS_3 / CFT_2 case can be considered as a “toy model”, where exact results are known.

For our purpose we consider the metric of ADS_3 in polar “coordinate”

$$ds^2 = \left[\left(\frac{r}{l} \right)^2 + 1 \right] dt^2 + \left[\left(\frac{r}{l} \right)^2 + 1 \right]^{-1} dr^2 + (rd\varphi)^2$$

where $\Lambda = -1/l^2$.

It is easy to show that the boundary ($r \rightarrow \infty$) is conformal; then the metric is invariant under the diffeomorphism which preserves the boundary conditions and their symmetries (i.e. the invariance group $SO(2, 2)$, or better the covering group $SL(2, C)$). Their infinitesimal generators define the classical Virasoro algebra.

Therefore it is useful to write the action in terms of a “would-be”

$$\begin{array}{ccc} \text{gauge field } A^{(\pm)} = & \omega & \pm \frac{1}{e} e & \text{so to give:} \\ & \downarrow & \downarrow & \\ & \text{spin conn.} & \text{vierbein} & \end{array}$$

$$I_{CS} = \frac{k}{4\pi} \int \text{Tr} \left\{ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\}$$

$I_{CS} \equiv$ Chern–Simon action; $F = dA + A \wedge \underline{A} = 0$ is a constraint

and k is the “central charge” of Kac–Moody algebra (I won’t discuss this aspect here).

BTZ Black Hole

A **rotating** Black Hole solution is described by metric (with Lorentz signature)

$$ds^2 = - \left[-M_L + \frac{r^2}{l^2} + \frac{J_L^2}{4r^2} \right] dt^2 \\ + \left[-M_L + \frac{r^2}{l^2} + \frac{J_L^2}{4r^2} \right]^{-1} dr^2 + r^2 \left[d\Phi - \frac{J_L}{2\pi} dt \right]^2$$

where J_L^2 is the angular momentum.

Notice that l , the curvature radius for ADS_3 : l sets the **length scale** for the problem.

As usual, we impose that the lapse function

$$N = -M_L + \frac{r^2}{l^2} + \frac{J_L^2}{4r^2}$$

is zero.

Then we get the value of the horizons radii r_{\pm} as

$$r_{\pm}^2 = \frac{M_L l^2}{r} \left[1 + \sqrt{1 - \frac{J_L^2}{M_L^2 \cdot l^2}} \right]$$

from which the invariants are

$$M_L^2 = \left(\frac{r_+^2 + r_-^2}{l^2} \right)^2 \quad M_L \text{ adimensional "mass" of BH}$$

$$J_L^2 = 4 \left(\frac{r_+ r_-}{l^2} \right)^2 \quad J_L \text{ angular momentum}$$

$$M_L^2 + J_L^2 = \frac{(r_+ + r_-)^4}{l^4}$$

For Euclidean space-time one gets a similar structure with the substitution

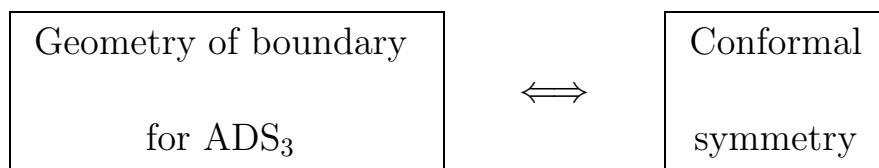
$$J_L^2 \longleftrightarrow -J^2$$

The dilatation symmetry generators L_0, \bar{L}_0 are deeply related to the invariants of BTZ Black Hole; in fact

$$M_L = L_0 + \bar{L}_0 \qquad J_L = L_0 - \bar{L}_0$$

Geometrically this identification follows from the form of the two Casimir invariants of $SO(2,2)$, explicitly written in function of the six Killing vectors.

We can see that



C) CFT₂ and Quantum effects

Brown–Henneaux long ago derived in a paper [*Comm. Math. Phys.* (1986)] a striking result for ADS₃ space–time

$$c = \frac{3}{2} \frac{l}{G_3}$$

– G_3 is the Newton constant for 3D

– c is the central charge of the “quantum” Virasoro algebra for CFT₂ (introduced in old time \simeq 1969 by Virasoro)

$$[L_n, L_m] = (n - m)L_{n+m} + c[n(n^2 - 1)]\delta_{n+m;0}$$

Simple derivation (Euclidean signature)

a) Modify $T_{\mu\nu}$ by a boundary term

$$\hat{T}_{\mu\nu}^E = T_{\mu\nu} - \frac{1}{8\pi G_3} \frac{1}{l} \gamma_{\mu\nu}$$

$\gamma_{\mu\nu}$ = boundary metric

The asymptotic ADS_3 metric is given by

$$ds^2 \underset{r^2 \rightarrow \infty}{=} \left(\frac{l}{r}\right)^2 (dr)^2 + \left(\frac{r}{l}\right)^2 [(dx_1)^2 + (dx_2)^2]$$

In complex coordinate $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$ we get the two sectors: left (right) correspond to analytical (antianalytical) z (\bar{z}) transformations; then the diffeomorphisms are the transformations

$$z \rightarrow z - f(z); \quad \bar{z} \rightarrow \bar{z} - g(\bar{z})$$

which preserve the boundary metric.

b) By requiring invariance of the metric we get for T_{zz} ($T_{\bar{z}\bar{z}}$)

$$T_{zz} \rightarrow T_{zz} + \dots + \frac{l^2}{2} (\partial_z^3 f(z)) dz^2$$

$$T_{\bar{z}\bar{z}} \rightarrow T_{\bar{z}\bar{z}} + \dots + \frac{l^2}{2} (\partial_{\bar{z}}^3 g(\bar{z})) d\bar{z}^2$$

One recognizes that the ∂_z^3 term generates a term analogous to the “central charge” term in the Virasoro algebra. That brings to

$$c = \frac{3}{2} \frac{l}{G_3}$$

But c is a “quantum” boundary effect in CFT₂, and $c/6$ measures the boundary energy (the Casimir energy).

Shift of vacuum energy	\Longleftrightarrow	Shift of vacuum energy
for ADS ₃ [Λ]		for CFT ₂ [$c/6$]

D) Black Hole thermodynamics for $D > 2$

Once identified the CFT_2 relevant for BTZ B.H. as a $c = 1$ “compactified” with $R^2 = 1$, one can derive T_H and S_H .

To understand how T_H is related to the topology of ADS_3 , by changing the coordinates in the hyperbolic 3-space H_3 , we find

$$\begin{aligned} ds^2 &= \frac{l^2}{z^2}(dx^2 + dy^2 + dz^2) = \\ &= \frac{l^2}{\sin^2 \chi} \left(\frac{dR^2}{R^2} + d\chi^2 + \cos^2 \chi d\vartheta^2 \right) \end{aligned}$$

ϑ, χ are angles, then the topology is $R_2 \times S_1$. In this coordinate system, the Euclidean time τ is periodic with period

$$\beta_H = \frac{1}{T_H} = \frac{2\pi r_+}{r_+^2 - r_-^2} l$$

from which for $r_- = 0$ (i.e. $J = 0$) one gets

$$S_H = \frac{1}{4} \frac{2\pi r_+}{G_3} \equiv \frac{1}{4} \frac{\text{Area}}{G_{\text{bulk}}}$$

as it should be.

One can show that S_H saturates the Bekenstein bound.

In CFT_2 all the previous periodicity is a generator of modular invariance

$$[\tau \rightarrow \tau + 1 \text{ transf.}]$$

Remind:

$$\tau \rightarrow \frac{1}{\tau}, \tau \rightarrow \tau + 1$$

generate the modular group $SL(2, Z)$, i.e. the seed of all the “dualities”.

E) Thermodynamics of the black holes for $D > 2$

1. The moving mirror case

I present an analysis of the moving mirror case (see C. Holzhey, F. Larsen, Frank Wilczek *Nucl. Phys. B* **424** (1994) 443-467; e-Print: hep-th/9403108, HLW from now on).

Indeed the moving mirror has been used to simulate the quantum behavior of fields near the horizon of the black holes.

Even in four dimensions they can be described by a two dimensional quantum field theory.

For that we exploit the conformal properties of the system.

Let's consider a moving mirror which obeys the equation of motion $x = f(t)$, $|\dot{f}| < 1$ and $f(t) = 0$ for $t < 0$.

We introduce also a massless scalar field described by the equation

$$\square\phi = \frac{\partial^2\Phi}{\partial u\partial v} = 0 \quad \text{where } u = x - t \text{ and } v = x + t$$

In a 2D Euclidean space the field equation is given by

$$\frac{\partial^2 \Phi}{\partial z \partial \bar{z}} = 0 \quad \text{where } z = x + it \quad \text{and} \quad \bar{z} = x - it$$

and at the boundary the reflection condition is

$$\phi(t, f(t)) = 0 \quad (\text{the reflection condition})$$

To describe the effects of a moving mirror on a scalar field, I consider its interaction with a detector through the Lagrangian

$$\mathcal{L}_{int.} = gk(\tau)\Phi[x(\tau)]$$

where g is a weak coupling constant and k is the monopole momentum of the detector.

The response function can be evaluated to be

$$\mathcal{F}(E) = \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' e^{-iE(\tau-\tau')} D^+ x(\tau) x(\tau')$$

which gives the counting of the number of particles found by the detector. The explicit evaluation by means of the Wightman function D^+ (or G^+ for a massive field), gives zero for a free field Φ .

The trajectory of an accelerated mirror modifies the form of the field in such a way that $\mathcal{F}(E)$ is no longer equal to zero, for the “reflected” field.

We choose the trajectory $z(t) \rightarrow -t - Ae^{-2\kappa t} + B$ as $t \rightarrow +\infty$ to simulate the behavior in the surroundings of a black hole. The mirror reflects only the null rays with trajectory $v < B$. **B acts like a horizon**, as all the rays with $v > B$ are not reflected.

For a detector moving with constant velocity w , it results the creation of a thermal bath of particles with

$$k_B T = \frac{\kappa}{2\pi} \left[\frac{1-w}{1+w} \right]^{1/2}$$

One can use this model to make clear aspects of the so called geometric entropy properties of a black hole [see HLW for details].

We notice that those results do not depend on D .

Indeed we are considering a one dimensional mirror, its properties are described in a two dimensional Minkowski spacetime.

The conformal properties of a massless scalar field in 2D become relevant in our case.

The trajectory

$$f(z) = D_1 + D_2 e^{z/4M}$$

is equivalent to the previous description.

This function is analogous to the coordinate transformation assigned in the Schwarzschild geometry, where the Hawking temperature is

$$T_H = 1/8\pi M$$

, for a four dimensional black hole with mass M .

Furthermore one can study the transformation properties of a hollow black hole described by the metrics

$$ds^2 = \begin{cases} dr^2 - d\tau^2 - r^2 d\Omega^2 & \text{if } \tau + r \leq V_s \\ \lambda^2 dt^2 - \lambda^{-2} dr^2 - r^2 d\Omega^2 & \text{if } \tau + r \geq V_s \end{cases}$$

The first metric describes the interior of the black hole (with null coordinates $U = \tau - r$ and $V = \tau + r$).

The second metric describes the exterior region (with null coordinates $u = \tau - r_*$ and $v = \tau + r_*$, where r_* is defined by the equation $dr_*/dr = 1/\lambda^2$).

In terms of these null coordinates we have

$$ds^2 = \begin{cases} dUdV - r^2 d\Omega^2 & \text{if } V \leq V_s \\ \lambda^2 dudv - r^2 d\Omega^2 & \text{if } v \geq V_s \end{cases}$$

It is easy to show that one can put $V(v) = v$. One can find $U(u)$ by demanding that along the worldline v_s of the shell r should agree in both systems.

The result is $U = c_1 + c_2 e^{-\kappa u}$, where $\kappa = 1/4M$ is the surface gravity.

2. Cardy formula and Casimir energy

Cardy (*Nucl. Phys. B* 1986) has proved that for CFT₂ modular invariance implies that the density of state is given by

$$U = 2\pi \sqrt{\frac{c}{6} \left(L_0 - \frac{c}{12} \right)}$$

where $c/12$ is the “vacuum” energy for $R(L)$ sector.

U **satisfies** the Bekenstein bound but at the same time it is related to the Casimir energy E_C **for any** D as

$$U = 2\pi R \sqrt{E_C(2E - E_C)}$$

$$E_C = \frac{c}{12}$$

Moreover the boundary quantum term $c/12$ does not depend on external temperature T .

3. Comments on entanglement entropy

For $D = 2$ it has been proved (see HLW) that

$$S_{ent} = \frac{c}{6} \ln \frac{L}{a}$$

where L and a are respectively the infrared and ultraviolet cutoffs.

Then it is easy to see that $\mathbf{S}_{ent} = \mathbf{S}_H = \mathbf{S}_{Bek}$, being the bound saturated both by S_H and S_{ent} .

On the other hand S_{ent} satisfies, on quite general ground, the area law, i.e.

$$S_{ent} = \frac{1}{4} \frac{2\pi r_+}{G_3}$$

Question: it is possible to extend those results for $D > 2$?

“There are good indications that say yes”

work in progress by G.M. and C.S.

Unitary representations of CFT_2

Ginsparg: hep/th/9108028 (long)

G. Maiella, C. Stornaiolo, *Int. Journ. of Mod. Phys. A* **22** (2007),
3429.

Appendix

Comments on the formula's (see pag. 5) from the **point of view** of CFT.

(M, J) are given by eigenvalues of dilatation operator $\Delta, \bar{\Delta}$.

For $c = 1$ theory compactified \hat{p}, \hat{w} are dual variables.

So if $\hat{p} = \frac{r_+}{l}$ $\hat{w} = \frac{r_-}{l}$, one gets

$$M = \Delta + \bar{\Delta} = \frac{r_+^2 + r_-^2}{l^2}$$

$$\frac{J}{l} = \Delta - \bar{\Delta} = 2\hat{p}\hat{w} = 2\frac{r_+r_-}{l^2}$$

“Duality” relations:

$$r_+ \leftrightarrow r_- \quad \text{or} \quad r_+ \leftrightarrow \frac{1}{l}$$

descend from $r_+r_- = \frac{1}{2}lJ$.